A SPHERICAL WAVE IN A VISCOELASTIC HALF-SPACE

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The propagation of spherical waves in an isotropic elastic medium has been studied sufficiently completely (see, e.g., [1-4]). It is proved [5, 6] that in imperfect solid media, the formation and propagation of waves similar to waves in elastic media are possible. With the use of asymptotic transform inversion methods in [7] a problem of an internal point source in a viscoelastic medium was investigated. The problem of an explosion in rocks in a half-space was considered in [8]. A numerical Laplace transform inversion, proposed by Bellman, is presented in [9] for the study of the action of an explosive pulse on the surface of a spherical cavity in a viscoelastic medium of Voigt type. In the present study we investigate the propagation of a spherical wave formed from the action of a pulsed load on the internal surface of a spherical cavity in a viscoelastic half-space. The potentials of the waves propagating in the medium are constructed in the form of series in special functions. In order to realize viscoelasticity we use a correspondence method [10]. The transform inversion is carried out by means of a representation of the potentials in integral form and subsequent use of asymptotic methods for their calculation. Thus, it becomes possible to investigate the behavior of a medium near the wave fronts. The radial stress is calculated on the surface of the cavity.

We are given a half-space with a spherical cavity of radius a at a depth h from the surface ($h > a$). At time $t = 0$ a pulsed load is applied to the surface of the cavity. We formulate the problem of finding the displacement field and the stress field in a half-space with account of the viscoelastic properties of the medium. The cylindrical r, z and spherical R, θ coordinate systems are connected, respectively, with the free surface of the half-space and the center of the cavity. We assume that the displacement field and stress field are independent of the azimuthal angle. The problem consists of solving the Cauchy equation

$$\sigma_{kl,l} = \rho \ddot{u}_k,$$

satisfying the conditions

$$u_k = 0,\ t < 0;\ \sigma_{kl} = f_1(R, t),\ R = a,\ t > 0;\ \sigma_{kl} = f_2(r, t),\ z = 0,\ t > 0,$$

where k = 1.2; l = 1.2. For account of energy dissipation in a viscoelastic medium for the oscillation of particles according to Hooke's law, additional terms are introduced by replacing the Lamé elastic constants λ and μ by some linear operators Λ and M or operators that are differential with respect to time with constant coefficients, or integral with respect to time with difference kernels.

Solution of the formulated unsteady problem by using a Laplace transformation with respect to time with parameter s in accordance with known relations reduces to finding the potential Φ of the longitudinal wave and the potential Ψ of the transverse waves from the equations

$$\begin{cases}(\nabla^2 - \alpha^2)\Phi = 0,\\ (\nabla^2 - \beta^2)\Psi = 0\end{cases} \tag{1}$$

with account of the boundary conditions

$$\text{for } R = a \begin{cases}\sigma_{RR} = -P,\\ \sigma_{R\theta} = 0;\end{cases} \tag{2}$$

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$$\text{for } z=0 \begin{cases}\sigma_{zz}=0,\\ \sigma_{rz}=0.\end{cases}$$

Here Φ, Ψ, and P are the Laplace transforms of the functions φ, ψ, and p; $f_1 = -P\delta(t)$; P = const;

$$\alpha = sv_p;\ \beta = sv_s;\ v_p = [\rho/(\Lambda + 2M)]^{1/2};\ v_s = (\rho/M)^{1/2}.$$

The unknown potentials are represented in the form

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \ldots = \sum_{m=0}^{\infty} \Phi_{2m} + \sum_{m=0}^{\infty} \Phi_{2m+1},$$
$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \ldots = \sum_{m=0}^{\infty} \Psi_{2m} + \sum_{m=0}^{\infty} \Psi_{2m+1}, \tag{3}$$

where Φ_0 and Ψ_0 are the potentials of the primary waves; Φ_{2m} and Ψ_{2m} are the potentials of the waves reflected from the spherical cavity; Φ_{2m+1} and Ψ_{2m+1} are the potentials of the waves reflected from the surface of the half-space. A solution is constructed for all the Φ_m and Ψ_m (m = 0, 1, 2, 3, ...) separately in the form of a series in spherical Hankel functions $h_n(z)$ of the first or second kind (this, in particular, makes it possible to represent the final solution in the form of combinations of elementary functions)

$$\Phi_m = \sum_{n=0}^{\infty} a_n^m h_n(i\alpha R_k) P_n(\cos\theta);$$
$$\Psi_m = \sum_{n=0}^{\infty} b_n^m h_n(i\beta R_k) \frac{\partial P_n(\cos\theta)}{\partial\theta}, \tag{4}$$

where the a_n^m and b_n^m are unknowns, determined from the boundary conditions; the $P_n(\cos\theta)$ are associated Legendre polynomials; k = 1.2; $R_1 = R$; $R_2 = r$. We limit ourselves to a search for the first three terms in (3). Considering Eq. (1) in a spherical coordinate system R, θ with origin at the center of the spherical cavity, and assuming in (4) that m = 0, $R_k = R_1 = R$, $h_n^{(1)}(i\alpha R)$ is a spherical Hankel function of the first kind, we obtain the potential of the primary wave:

$$\Psi_0 = 0,\ \Phi_0 = -\frac{P}{A(a)} h_0^{(1)}(i\alpha R).$$

The unknown A(a) is determined from the boundary condition (2)

$$A(a) = -(M/\alpha a^3)(\beta^2 a^2 + 4a\alpha + 4).$$

Finally,

$$\Phi_0 = (Pa^3/RM)[e^{-\alpha(R-a)}/(\beta^2 a^2 + 4a\alpha + 4)]. \tag{5}$$

To determine the potentials

$$\Phi_1 = \sum_{n=0}^{\infty} a_n^1 h_n^{(2)}(i\alpha r) P_n(\cos\theta),$$
$$\Psi_1 = \sum_{n=0}^{\infty} b_n^1 h_n^{(2)}(i\beta r) \frac{\partial P_n(\cos\theta)}{\partial\theta}$$

of the waves reflected from the surface of the half-space, we use an integral Sommerfeld representation of spherical functions [4]:

$$h_0^{(2)}(icR) = \frac{1}{c}\int_0^{\infty} e^{\sqrt{k^2+c^2}} \frac{J_0(kr)}{\sqrt{k^2+c^2}} k\,dk,$$

which allows us to write a solution of Eqs. (1) in the cylindrical coordinate system r, z, coupled with the surface of the half-space, in the form

$$\Phi_{2m+1} = \int_0^{\infty} A(k) \frac{J_0(kr)\,k}{\sqrt{k^2+\alpha^2}} e^{-\sqrt{k^2-\alpha^2}}\,dk;$$
$$\Psi_{2m+1} = \int_0^{\infty} B(k) \frac{J_1(kr)\,k}{\sqrt{k^2+\beta^2}} e^{-\sqrt{k^2+\beta^2}}\,dk.$$

The condition for vanishing of the stresses that arise due to the potentials $\Phi_{2m} + \Phi_{2m+1}$ and $\Psi_{2m} + \Psi_{2m+1}$ for z = 0 leads to a system of equations for finding the unknowns A(k) and B(k). Finally, the potentials of the once-reflected waves acquire the form

$$\Phi_1 = \frac{P}{\alpha A(a)} \int_0^{\infty} \frac{T(k)\,k}{R(k)\sqrt{k^2+\alpha^2}} J_0(kr)\, e^{-(h-z)\sqrt{k^2+\alpha^2}}\,dk;$$

$$\Psi_1 = -\frac{P}{\alpha A(a)}\int_0^\infty \frac{4(2k^2+\beta^2)k^2}{R(k)} J_1(kr)\, e^{-h\sqrt{k^2+\alpha^2}+z\sqrt{k^2+\beta^2}}\, dk,$$

where $z < h$; $R(k) = (2k^2+\beta^2)^2 - 4k^2\sqrt{k^2+\alpha^2}\sqrt{k^2+\beta^2}$; $T(k) = (2k^2+\beta^2)^2 + 4k^2\sqrt{k^2+\alpha^2}\sqrt{k^2+\beta^2}$. The transform inversion is simplified with the substitution of the variable of integration

$$k = sv_p p. \tag{6}$$

Converting from Bessel functions to Hankel functions and using the asymptotic representation of the latter, the integrals can be calculated from the method of steepest descents:

$$\Phi_1 = \frac{2PF_p e^{-i\frac{\pi}{4} - sv_p R'}}{\alpha A(a, s) R'};\ \Psi_1 = \frac{2PF_s e^{-i\frac{3\pi}{4} - sv_p d}}{\alpha A(a, s)\sqrt{r}}, \tag{7}$$

where

$$R' = [r^2 + (h-z)^2]^{1/2};\ d = h\sec\alpha_1 - z\nu\sec\alpha_2;\ \nu = v_p/v_s;$$

$$F_p = \frac{(2r^2 - \nu^2 R'^2) + 4r^2 R'(h-z)\sqrt{R'^2\nu^2 - r^2}}{(2r^2 - \nu^2 R'^2)^2 - 4r^2 R'(h-z)\sqrt{R'^2\nu^2 - r^2}}; \tag{8}$$

$$F_s = \frac{\nu^{3/2}\sin^2\alpha_2\cos 2\alpha_2}{(\nu\cos^2 2\alpha_2 + 4\sin^2\alpha_2\cos\alpha_1\cos\alpha_2)\sqrt{\nu h\sec^3\alpha_1 - z\sec^3\alpha_2}},$$

where the following relations are satisfied:

$$\sin\alpha_1/v_p = \sin\alpha_2/v_s,\ r = h\,\mathrm{tg}\,\alpha_1 - z\,\mathrm{tg}\,\alpha_2$$

(α_1 is the angle of incidence on the boundary of the half-space of the longitudinal wave; α_2 is the angle of reflection of the transverse wave that is formed).

Considering Eqs. (1) in the coordinate system coupled with the center of the cavity for $m = 2$ in (4) the potentials Φ_2 and Ψ_2 of the twice-reflected waves are represented in the form

$$\Phi_2 = \sum_{n=0}^\infty a_n^2 h_n^{(1)}(i\alpha R) P_n(\cos\theta);\ \Psi_2 = \sum_{n=0}^\infty b_n^2 h_n^{(1)}(i\beta R)\frac{\partial P_n(\cos\theta)}{\partial\theta}.$$

The unknowns a_n^2 and b_n^2 can be expressed in terms of a_n^1 and b_n^1, taking into account the vanishing of the stresses that arise due to the potentials $\Phi_{2m+2} + \Phi_{2m+1}$ and $\Psi_{2m+2} + \Psi_{2m+1}$ for $R = a$, i.e., on the surface of the cavity. This condition leads to the system of equations

$$\begin{cases} A_n(a)\, a_n^{2m+2} + B_n(a)\, b_n^{2m+2} = -\left[E_n(a)\, a_n^{2m+1} + F_n(a)\, b_n^{2m+1}\right], \\ C_n(a)\, a_n^{2m+2} + D_n(a)\, b_n^{2m+2} = -\left[G_n(a)\, a_n^{2m+1} + L_n(a)\, b_n^{2m+1}\right]. \end{cases}$$

The coefficients E_n, F_n, G_n, and L_n coincide with A_n, B_n, C_n, and D_n, respectively, when in the latter, instead of spherical functions of the first kind we take spherical functions of the second kind. The system is solved for $m = 2$ with respect to the unknowns a_n^2 and b_n^2. Thus,

$$\Phi_2 = \Phi_{22} + \Psi_{21} = \sum_{n=0}^\infty a_n^1 \frac{B_n G_n - D_n E_n}{\Delta} h_n^{(1)}(i\alpha R) P_n(\cos\theta) + \sum_{n=0}^\infty b_n^1 \frac{B_n L_n - F_n D_n}{\Delta} h_n^{(1)}(i\alpha R) P_n(\cos\theta);$$

$$\Psi_2 = \Phi_{21} + \Psi_{22} = \sum_{n=0}^\infty \frac{A_n G_n - C_n E_n}{\Delta} a_n^1 h_n^{(1)}(i\beta R)\frac{\partial P_n(\cos\theta)}{\partial\theta} + \sum_{n=0}^\infty \frac{G_n F_n - A_n L_n}{\Delta} b_n^1 h_n^{(1)}(i\beta R)\frac{\partial P_n(\cos\theta)}{\partial\theta},$$

where

$$\Delta = A_n D_n = B_n C_n;$$

$$a_n^1 = \frac{Pi^n(2n-1)}{\alpha A(a)}\int_0^\infty \frac{T(k)\,k}{R(k)\sqrt{k^2+\alpha^2}} P_n\left(\frac{\sqrt{k^2+\alpha^2}}{\alpha}\right) e^{-2h\sqrt{k^2+\alpha^2}} dk; \tag{9}$$

$$b_n^1 = -\frac{Pi^{n-2}(2n+1)}{\alpha\beta n(n+1)A(a)}\int_0^\infty \frac{4(2k^2+\beta^2)k^3}{R(k)} P_n'\left(\frac{\sqrt{k^2+\beta^2}}{\beta}\right) e^{-h(\sqrt{k^2+\alpha^2}+\sqrt{k^2+\beta^2})} dk;$$

Φ_{22} and Ψ_{21} are the potentials of the reflected longitudinal and transverse waves that arise from the incidence of the longitudinal wave; Φ_{21} and Ψ_{22} are the potentials of the reflected longitudinal and transverse waves that arise from the incidence of the transverse wave. For convenience the inversions of the potentials Φ_2 and Ψ_2 are represented in integral form. For this we can use the asymptotic representation of the spherical functions, the integral representation of Legendre polynomials, and the Watson method of representation of series in the form of integrals. With account of the earlier introduced substitution (6) and Eqs. (9), the potential, for example, of the twice-reflected longitudinal wave, takes the form

$$\Phi_{22} = -\frac{Pe^{i\frac{\pi}{4}}}{(2\pi)^2 \alpha A(a)} \int\limits_{\mathscr{L}_n} \int\limits_{-\infty}^{\infty} \int\limits_{C_\xi} \int\limits_{C_\eta} F(p, n)\, e^{f} dn dp d\xi d\eta, \tag{10}$$

where

$$f = f(s, p, n, \xi, \eta) = -sv_p\left(2a + R + 2h\sqrt{p^2+1}\right) - i7\pi n/4 +$$
$$+ n \ln\left[\left(\sqrt{p^2+1} - p\cos\xi\right)\left(-\cos\theta - i\sin\theta\cos\eta\right)\right];$$
$$F(p, n) = \frac{T(p)\, pi^{n+1}(2n+1)}{2R(p)\sqrt{p^2+1}\sin n\pi};$$

$\mathscr{L}_n$ is the contour of integration in the complex n place, consisting of a loop enclosing the positive part of the real axis from the point Re (n) < 1/2; C_ξ and C_η are closed contours of integration, respectively, in the ξ and η complex planes, containing, respectively, the points $\xi = 0$ and $\eta = 0$. The integrals in (10) can be calculated, e.g., by the method of steepest descents. The potentials Ψ_{21}, Φ_{21}, and Ψ_{22}, on the basis of the same remarks, can be represented in similar form and can be calculated from the method of steepest descents.

The inversion of the obtained transform is carried out under the assumption that a viscoelastic medium has an instantaneous elasticity and

$$v_p/v_s = \nu = \text{const}.$$

According to [10], the quantity v_p can be replaced by $v_p(s) = [\rho I(s)]^{1/2}$, where $I(s)$ for a rather wide class of viscoelastic materials can be represented in the form

$$I(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}. \tag{11}$$

With the use of asymptotic methods, phenomena at the fronts of a strong discontinuity can be studied [6]. In this case we study the asymptotic behavior of the solution for $s \to \infty$, and $v_p(s)$ can be represented in the form

$$v_p(s) = [\rho I(s)]^{1/2} = \varkappa^{-1}\left[1 + \frac{A}{2s} + \left(B - \frac{A^2}{4}\right)\frac{1}{2s^2} + \left(C - \frac{AB}{2} + \frac{A^3}{8}\right)\frac{1}{2s^3} + 0(s^{-4})\right],$$

where $\varkappa = (\rho a_m/b_m)^{-1/2}$ is the wave-propagation velocity; the coefficients A, B, and C are determined by the expansion of (11) in a series in negative powers of s. For the transform inversion we must take into account that only the expression

$$e^{-sv_p f}/\alpha A(a, s) \tag{12}$$

depends on the transformation parameter s, where f is a function that is independent of s and takes a specific form in each case. We limit ourselves to the inversion of the potentials of the initial and once-reflected waves. The expansion of (12) in a series and the use of a known inversion formula leads to the equation

$$\varphi_0 = \frac{Pa^3}{\rho R 2\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{\exp\left\{-(R-a)\left(\frac{s}{\varkappa} + \gamma\right) + st\right\}}{a^2s^2 + 4a^2\nu^2\varkappa^2 s + 4\nu^2\varkappa^2 + 2a\nu^2\varkappa A}\left[1 - \left(\frac{\varepsilon_1}{\varepsilon_0} + \frac{B}{2\varkappa} - \frac{A^2}{8\varkappa}\right)\frac{1}{s} + 0(s^{-2})\right] ds. \tag{13}$$

The representation of the potentials φ_1 and Ψ_1, reflected from the surface of the half-space of the waves [which can be noted from a comparison of (5) and (7)] is similar in form.

Assuming in (11) that $a_2 \neq 0$, $b_2 \neq 0$, $b_0 \neq 0$, and all the remaining a_n and b_n equal zero, we have $\gamma = 0$, $A = 0$, $B = -b_0/2b_2$, $\epsilon_1 = -(2a\nu^2/\varkappa)b_0/b_4$, $\epsilon_0 = a^2s^2 + 4a\nu^2\varkappa s + 4\varkappa^2\nu^2$.

According to the Jordan lemma, the integral in (13) is nonzero for $t \geq (R - a)/\varkappa$, and equals the sum of the residues at the singular points of the integrand function. Hence, the potentials will be nonzero only from that moment of time when the front of the appropriate wave arrives at the point being considered. Calculations for the first two terms of the series in the dimensionless parameters $t\varkappa/a = \tau$, $sa/\varkappa = \xi$, $R/a = R_0$ lead to the following expressions: for the potential of the primary wave

$$\frac{\varphi_0}{aP} = \frac{1}{2R_0}\left\{\frac{1}{\nu\sqrt{1-\nu^2}} e^{-2\nu^2(\tau - R_0 + 1)}\sin\gamma_0 + \frac{b_0 a_2}{4b_2^2\nu^2}\left[1 - e^{-2\nu^2(\tau - R_0 + 1)}\left(\cos\gamma_0 - \frac{\nu}{\sqrt{1-\nu^2}}\sin\gamma_0\right)\right]\right\},$$

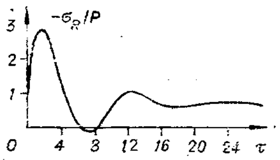

Fig. 1

where $\gamma_0 = 2\nu\sqrt{1-\nu^2}(\tau - R_0 + 1)$; for the potentials of the once-reflected waves

$$\frac{\varphi_1}{aP} - \frac{\sqrt{2}F_p}{2R_1}\left\{\frac{e^{-2\nu^2(\tau-R_1+1)}}{\nu\sqrt{1-\nu^2}}\sin\gamma_1 + \frac{b_0 a_2}{8b_2^2\nu^2}\left[1 - e^{-2\nu^2(\tau-R_1+1)}\left(\cos\gamma_1 - \frac{\nu}{\sqrt{1-\nu^2}}\sin\gamma_1\right)\right]\right\};$$

$$\frac{\psi_1}{aP} = -\frac{\sqrt{2}F_s}{2\sqrt{r_1}}\left\{\frac{e^{-2\nu^2(\tau-d+1)}}{\nu\sqrt{1-\nu^2}}\sin\gamma_2 + \frac{b_0 a_2}{8b_2^2\nu^2}\left[1 - e^{-2\nu^2(\tau-d+1)}\left(\cos\gamma_2 - \frac{\nu}{\sqrt{1-\nu^2}}\sin\gamma_2\right)\right]\right\},$$

where d, F_p, and F_S are taken from (8) with r, R', h, and z replaced by the dimensionless parameters $r_1 = r/a$, $R_1 = R'/a$, $H = h/a$, and $Z = z/a$; $\gamma_1 = 2\nu\sqrt{1-\nu^2}(\tau - R_1 + 1)$; $\gamma_2 = 2\nu\sqrt{1-\nu^2}(\tau - d + 1)$. Based on known formulas we can calculate the displacements and the stresses at points of the half-space. For example, the radial stress arising from the propagation of the primary wave is calculated from the equation

$$-\frac{\sigma_R}{P} = \frac{1}{R_0^3}\left\langle e^{-2\nu^2(\tau-R_0+1)}\left[\frac{(1-\nu)\left(1+R_0-4\nu R_0^2\right)+2\nu(1-2\nu)}{2(1-2\nu)}\cos\gamma_0 + \frac{(1-2\nu^2R_0)^2}{\nu\sqrt{1-\nu^2}}\sin\gamma_0\right] + \frac{b_0a_2}{4b_2^2}\left\{\frac{1-4\nu^2}{4\nu^2} + \right.\right.$$
$$\left.\left. + e^{-2\nu^2(\tau-R_0+1)}\left[\frac{\nu(1-4R_0\nu^2)-(1-\nu)(1+2\nu)\left(1-R_0^2\nu\right)}{4\nu^2(1-2\nu)}\cos\gamma_0 + \frac{(1-\nu)(1-2R_0\nu^2)^2-\nu(1-2R_0)}{4\nu^2\sqrt{1-\nu^2}}\sin\gamma_0\right]\right\}\right\rangle.$$

For $R_0 = 1$, $\tau = 0$, the equation gives the pressure on the surface of the cavity at the initial time $\sigma_R = -P$. We determine the variation of the stress on the surface of the cavity up to the time of arrival of the reflected wave for $\nu = 0.3$; $R_0 = 1$; $H = 15$; $0 \le \tau < 28$; $b_0a_2/b_2^2 = 1$. The calculation results are shown in Fig. 1.

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